International Journal of Applied Mathematics & Statistical Sciences (IJAMSS)
ISSN (P): 2319–3972; ISSN (E): 2319–3980
Vol. 11, Issue 2, Jul–Dec 2022; 43–50

International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations

FUNCTIONS OF BOUNDED RADIUS ROTATION OF ANALYTIC FUNCTIONS

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ABSTRACT

The classes $V_k(\alpha)$, $R_k(\alpha)$, $V_{k1}(\alpha)$ and $R_{k1}(\alpha)$ of analytic function, We establish a relation between the functions of bounded boundary and bounded radius rotations.

KEYWORDS: Analytic, Analytic Functions with Bounded Radius and Boundary Rotations, Starlike, Convex, Subordination

MATHEMATICS SUBJECT CLASSIFICATION: **30C45**

Article History

Received: 16 Aug 2022 | Revised: 17 Aug 2022 | Accepted: 24 Aug 2022

1. INTRODUCTION

Let A be the class of functions f of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \tag{1.1}$$

Which are analytic in the unit disc E = {z: |z| < 1}.

The definition of subordination is $f\in$ A is subordinate to g $\in$ A, written as $f\prec g$,there exists Schwarz function w(z) with w(0) = 0 and |w(z)| < 1 ($z\in$ E) such that f(z) = g(w(z)), In particular, when g is univalent, then the above subordination is equivalent to f(0)’ = g(0) and f(E) $\subseteq$ g(E), for any two analytic functions

$$f(z)=\sum_{n=0}^{\infty}a_n z^n,\quad g(z)=\sum_{n=0}^{\infty}b_n z^n (z\in E). \tag{1.2}$$

then convolution

$$(f*g)(z)=\sum_{n=0}^{\infty}a_n b_n z^n \quad (\mathrm{z}\in \mathrm{E}) \tag{1.3}$$

We denote by $s^*(\alpha)$, $c(\alpha)$, $(0\le\alpha<1)$ the classes of starlike and convex functions of order α,

respectively defined by

$$\begin{aligned} s^*(\alpha) &= \left\{ f \in A : \operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha, \ \ z \in E \right\} \\ c(\alpha) &= \left\{ f \in A ; \ zf'(z) \in s^*(\alpha), \ \ z \in E \right\} \end{aligned} \tag{1.4}$$

For $\alpha = 0$, we have the well know classes of starlike and convex univalent functions denoted s^* and c, respectively.

Let $P_k(\alpha)$ be the class of functions P(z) analytic in the unit disc E satisfying the properties P(0)=1 and

$$\int_0^{2\pi} \left| \operatorname{Re}\frac{P(z)-\alpha}{1-\alpha} \right| d\theta \le k\pi, \tag{1.5}$$

Where $z = re^{i\theta}$, $k \ge 2,$ and $0 \le \alpha < 1$, using Herglotz – Stieltijes formula

$$P(z) = \left(\frac{k}{4} + \frac{1}{2}\right) P_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) P_2(z), \qquad z \in E \tag{1.6}$$

Where $P(\alpha)$ is the class of functions with real part greater then α and $P_i \in P(\alpha),$ for i = 1, 2 we define the following classes

$$\begin{aligned} R_k(\alpha) &= \left\{ f : f \in A\, and \, \frac{\left[\lambda z^2 f''(z) + zf'(z)\right]}{\lambda z(f'(z) - f'(-z)) + (1-\lambda)(f(z) - f(-z))} \in P_k(\alpha), 0 \le \lambda < 1 \right\}, \quad t > 0 \\ V_k(\alpha) &= \left\{ f : f \in A\, and \, \frac{\left[(\lambda z^2 f''(z) + zf'(z)\right]'}{\lambda z(f''(z) - f''(-z)) + \left[(1-\lambda)(f'(z) - f'(-z)\right]'} \in P_k(\alpha), 0 \le \lambda < 1 \right\}, \quad t > 0 \end{aligned} \tag{1.7}$$

$$R_{k1}(\alpha) = \left\{ f : f \in A\, and \, \frac{\left[\gamma\lambda z^3 f'''(z) + (2\gamma\lambda + \gamma + \lambda) z^2 f''(z) + zf'(z)\right]}{\begin{array}{l}\gamma\lambda z^2 (f''(z) - f''(-z)) + (\lambda - \gamma) z (f'(z) - f'(-z) \\ + (1 - \lambda + \gamma)(f(z) - f(-z))\end{array}} \in P_k(\alpha), \right\} \tag{1.8}$$
$$0 \le \lambda \le \gamma < 1.$$

$$V_{k1}(\alpha) = \left\{ f : f \in A\, and \, \frac{\left[\gamma\lambda z^3 f'''(z) + (2\gamma\lambda + \gamma + \lambda) z^2 f''(z) + zf'(z)\right]'}{\begin{array}{l}\gamma\lambda z^2 (f''(z) - f''(-z)) + (\lambda - \gamma) z (f'(z) - f'(-z) \\ + \left[(1 - \lambda + \gamma)(f(z) - f(-z)\right]'\end{array}} \in P_k(\alpha), \right\} \tag{1.9}$$
$$0 \le \lambda \le \gamma < 1.$$

We note that

$$f \in V_k(\lambda) \Leftrightarrow zf' \in R_k(\lambda) \qquad f \in V_{k1}(\lambda) \Leftrightarrow zf' \in R_{k1}(\lambda) \tag{1.10}$$

For $\lambda = 0, t = 0, \gamma = 0$ we obtain the well known classes R_k, R_{k1}, V_k, V_{k1} of analytic functions with bounded radius and bounded boundary rotations, respectively. These classes are studied by Noor [3] in more details, also it can easily by seen $R_k{}^2(\alpha) = S^*(\alpha)$ and $V_2(\alpha) = c(\alpha)$. Goel [6] proved that $f \in c(\alpha)$ implies that $f \in s^*(\alpha)$ and $f \in S^*(\alpha)$

Where $$\beta = \beta(\alpha) = \begin{cases} \dfrac{4^{\alpha}(1-2\alpha)}{4-2^{2\alpha+1}} & , \alpha \neq \dfrac{1}{2} \\ \dfrac{1}{2}\ln 2 & , \alpha = \dfrac{1}{2} \end{cases}$$

and the result is sharp. In this paper, we prove the result of Goel [6] for the classes $V_k(\alpha)$, $R_k(\alpha)$, $V_{k1}(\alpha)$ and $R_{k1}(\alpha)$ by using convolution and Subordination techniques.

2. PRELIMINARY RESULTS

We need the following results to obtain our results.

Lemma 2.1:

Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ and $\Psi(u,v)$ be a complex valued function satisfying the conditions

(i) $\Psi(u,v)$ is continuous in a domain $D \subset C^2$.

(ii)$(1,0) \in D$ and $\operatorname{Re}\psi(1,0) > 0,$ when ever $(iu_2, v_1) \le 0$ when ever $(iu_2, v_1) \in D$ and $v_1 \le -\left(\frac{1}{2}\right)\left(1 + u_2^2\right)$

If $h(z) = 1 + c_1 z + \ldots\ldots$ is a function analytic in E. Such that $(h(z), zh'(z)) \in D$ and $\operatorname{Re}\psi(h(z), zh'(z)) > 0$ for $z \in E,$ then Re h(z)>0 in E.

3. MAIN RESULTS

Theorem 3.1

Let $f \in V_k(\alpha),$ then $f \in R_k(\alpha),$ where

$$\beta = \frac{1}{4}(2\alpha - (1+\lambda) + 2\alpha\lambda) + \sqrt{\begin{array}{l}(1+\lambda)^2 + 4\alpha^2 + 4\alpha^2\lambda^2 - 4\alpha(1+\lambda) + 8\alpha^2\lambda - 4\alpha(1+\lambda) \\ + 8 - 8\lambda + 8\lambda + (1+\lambda^2).\end{array}}$$

Proof:

Let

$$\frac{(\lambda z^2 f''(z) + zf'(z))}{\lambda z(f(z) - f(-z)) + (1-\lambda)(f(z) - f(-z))} = (1-\beta)P(z) + \beta \tag{3.1}$$

$$=(1-\beta)\left[\left(\frac{k}{4}+\frac{1}{2}\right)P_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)P_2(z)\right]+\beta \tag{3.2}$$

P(z) is analytic in E with P(0)=1 then

$$\frac{\left[\left[\lambda z^2 f''(z)+z f'(z)\right)\right]'}{\lambda z\left[f'(z)-f'(-z)\right]+(1-\lambda)\left[f(z)-f(-z)\right]}=(1-\beta)p(z)+\beta$$
$$+\frac{\begin{bmatrix}\lambda z^2 f''(z)+ztf'(z)+(1-\beta)p'(z)\\ +(1-\lambda)z+\lambda zf'(-z)\end{bmatrix}}{\lambda z\left[f'(z)-f'(-z)\right]+(1-\lambda)(1-\beta)p(z)+\beta} \tag{3.3}$$

$$\frac{1}{(1-\alpha)}\left[\frac{\left[\left[\lambda z^2 f''(z)+z f'(z)\right)\right]'}{\lambda z\left[f'(z)-f'(-z)\right]+(1-\lambda)\left[f(z)-f(-z)\right]}-\alpha\right]$$
$$=\frac{1}{(1-\alpha)}\left[(1-\beta)p(z)+\beta-\alpha+\frac{\left[\lambda z^2 f''(z)+z\lambda f'(z)+(1-\lambda)z\right].(1-\beta)p'(z)}{\lambda z\left[f'(z)-f'(-z)\right]+(1-\lambda)\left[(1-\beta)p(z)+\beta\right]}\right] \tag{3.4}$$
$$=\frac{\beta-\alpha}{1-\alpha}+\frac{1-\beta}{1-\alpha}\left[p(z)+\left(\frac{\left(\frac{1}{(1-\beta)}\right)(L+M)p'(z)}{(N+Q)(p(z)+\frac{\beta}{1-\beta}}\right)\right]$$

Where

$$L=\lambda z^2 f''(z)$$

$$M=\lambda zf'(z)+(1-\lambda)z$$

$$N=\lambda z\left[f'(z)-f'(tz)\right]$$

$$Q=(1-\lambda)$$

Since $f\in V_k(\alpha)$, it implies that

$$=\frac{\beta-\alpha}{1-\alpha}+\frac{1-\beta}{1-\alpha}\left[p(z)+\frac{\left(\frac{1}{1-\beta}\right)\left(L+M\right)p'(z)}{\left(N+Q\right)\left(p(z)+\frac{\beta}{(1-\beta}\right)}\right]\in p_{k,}\qquad z\in E \tag{3.5}$$

We define

$$\varphi_{a,b}(z)=\frac{1}{1+b}\frac{z}{(1-z)\alpha}+\frac{b}{1+b}\frac{z}{(1-z)^{1+a}} \tag{3.6}$$

with $a=\dfrac{1}{1-\beta}$, $b=\dfrac{\beta}{1-\beta}$ by using (3.1) with convolution, see [5], we have, that

$$\frac{\varphi_{a,b}(z)}{z} * p(z) = \left(\frac{k}{4}+\frac{1}{2}\right)\left[\frac{\varphi_{a,b}(z)}{z} * p_1(z)\right] - \left(\frac{k}{4}-\frac{1}{2}\right)\left[\frac{\varphi_{a,b}(z)}{z} * p_2(z)\right]$$

Implies

$$p(z) + \frac{a(L+M)p'(z)}{(N+Q-R)(p(z)+b)} = \left(\frac{k}{4}+\frac{1}{2}\right)\left(p_1(z) + \frac{a(L+M)p_1'(z)}{(N+Q)(p_1(z)+b}\right) - \left(\frac{k}{4}+\frac{1}{2}\right)\left(p_2(z) + \frac{a(L+M)p_2'(z)}{(N+Q)(p_2(z)+b}\right) \tag{3.7}$$

Thus from (3.5) and (3.7) we have

$$\frac{\beta-\alpha}{1-\alpha} + \frac{(1-\beta)}{(1-\alpha)}\left[p_i(z) + \frac{a(L+M)p_i'(z)}{(N+Q)(p_i(z)+b}\right] \in p \quad \mathrm{i}=1,2 \tag{3.8}$$

We now from functional $\psi(u,v)$ by choosing $u = p_i(z),\ v = zp_i'(z)$ in (3.8) and note that the first two conditions of Lemma 2.1 are likely satisfied, we check the condition as follows

$$\begin{aligned}
\mathrm{Re}[\psi(iu_2,v_1)] &= \frac{1}{1-\alpha}\left[(\beta-\alpha) + \mathrm{Re}\left(\frac{\lambda z^2 f''(z)p_i'(z) + f'(z)v_1}{\left(iu_2+\frac{\beta}{1-\beta}\right)[\lambda z(f''(z)-f''(tz)] + (1-\lambda)[f'(z)-f'(z)]'}\right)\right] \\
&= \frac{1}{1-\alpha}\left[(\beta-\alpha) + \frac{\left(\lambda z^2 f''(z)p_i'(z) + f'(z)v_1\right)\left(\frac{\beta}{(1-\beta}\right)}{\left(iu_2+\frac{\beta}{1-\beta}\right)\begin{pmatrix}\lambda z\left(f''(z)+f'(z)\right)+(1-\lambda) \\ \left(f(z)-f(z)\right)\end{pmatrix}'}\right] \\
&\le \frac{1}{1-\alpha}\left((\beta-\alpha) - \frac{\frac{1}{2}\left((1+u_2^2)f'(z)+\lambda z^2 f''(z)\right)\left(\frac{\beta}{1-\beta}\right)}{\left(u_2^2+\left(\frac{\beta}{1-\beta}\right)^2\right)\left(\left(f(z)-f(z)\right)+\lambda z^2 f''(z)\right)}\right) \\
&= \frac{2(\beta-\alpha) - \left((1+u_2^2)f'(z)+\lambda z^2 f''(z)\right)\left(\frac{\beta}{1-\beta}\right)}{2\left(u_2^2+\left(\frac{\beta}{1-\beta}\right)^2\right)\left((f(z)-f(z)+\lambda z^2 f''(z)\right)}
\end{aligned} \tag{3.9}$$

$$= \frac{A+Bu_2^2}{2c}, \qquad 2C > 0,$$

Where

$$A=\frac{\beta}{(1-\beta)^2}\left[2(\beta-\alpha)\beta[(f(z)-f(z)]+\ \lambda z^2\ f''(z)\,]'\ -\ [\,f'(z)\ +\lambda z^2\ f''(z)\,]\,(1-\beta)\right]$$

$$B=\frac{1}{(1-\beta)}\,[2(\beta-\alpha)(1-\beta)\,[(f(z)-f(z)]+\ \lambda z^2\ f''(z)\,]''\ -\beta\ [\,f'(z)\ +\lambda z^2\ f''(z)\,] \tag{3.10}$$

$$C=\left[u_2^2+\left(\frac{\beta}{1-\beta}\right)^2\right]\ (1-\alpha)\,[(f(z)-f(z)]+(1+\lambda)\,\lambda z^2\ f''(z)\,]'$$

The right hand side of (3.9) is negative if $A\le 0$ we have

$$\beta=\beta(\alpha)=\frac{1}{4}(2\alpha-(1+\lambda)+2\alpha\lambda)+\sqrt{\begin{matrix}(1+\lambda)^2+4\alpha^2+4\alpha^2\lambda^2-4\alpha(1+\lambda)+8\alpha^2\lambda-4\alpha(1+\lambda)\\ +8-8\lambda+8\lambda+(1+\lambda^2).\end{matrix}}$$

Theorem 3.2

Let $f\in V_{k1}(\alpha)$, then $f\in R_{k1}(\alpha)$, where

$$\beta=\frac{1}{4}(2\alpha-(1+\lambda)(1+\gamma)+2\alpha\gamma\lambda+\sqrt{\begin{matrix}(1+\lambda+\gamma)^2+4\alpha^2\lambda^2\gamma+6\alpha^2\gamma^2\lambda^2-4\alpha(1+\gamma)(1+\lambda)\\ +16\alpha^2\lambda\gamma-4\alpha t(1+\lambda)(1+\gamma)+8(1+\lambda)(1+\gamma)-12\gamma\lambda\\ +(1+\gamma^2)(1+\lambda^2).\end{matrix}}$$

REFERENCE

1. *B.Pinchuk, "Functions of bounded boundary rotation," Israel journal of Mathematics, vol. 10, no. 1, pp.6-16, 1971.*

2. *K.S.Padmanabhan and R.Parvathan, "Properties of a class of functions with bounded boundary rotation," AnnalesPolomiciMathematici, vol.31, no.3, pp.311-323, 1975.*

3. *K.I. Noor, "Some properties of certain analytic function,"journal of natural geometry, vol .7, no.1, pp.11-20, 1995*

4. *K.I.Noor, "on some subclassesof function with bounded radius and bounded boundary rotation,"panamerican mathematical journal, vol.6, no.1, pp.75-81, 1996*

5. *K.L.Noor, "on analytic function related to certain family of integral operators,"journal of inequalities in pure and applied mathematics, vol.7, no.2, article 69, 6pages, 2006.*

6. *R.N.Goel, "functions starlike and convex of order α ," Journal of the London Mathematical society.Second Series, vol.s2-9, no.1, pp.128-130, 1974*

7. *D.A.Brannan, "on functions of bounded boundary rotation .I," proceedings of the Edinburgh mathematical society .series II, vol.16, pp.339-347, 1969*

8. *D.Pinchuk, "on starlike and convex function order a," duke mathematical journal, vol.35, pp.721-734, 1968*

9. *S.S.Miller, "Differential in equalities and caratheodoryfunctions,"bulletin of the American mathematical society,*

vol.81, pp.79-81, 1975.

10. *S.S.Miller and P.T.Mocanu, "differential subordinations :theory and application ,vol.225of monographs and text bookes in pure and applied mathematicals," marcel dekker ,new york ,NY,USA,2000*
11. *Z.Nahari conformal mappings, Dover, New [11] Z.Nahari conformal mappings, Dover, NewYork, NY, USA, 1952*
12. *I.S.Jack, "functions starlike and convex of order a,"journal of the London mathematical society .second series, vol.3, pp.469-474, 1971*
13. *T.H.MacGregor, "A subordination for convex function of order a," journal of the London mathematical society second series, vol.9, pp.530-536, 1975.*